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Threshold Resonances and Eigenvalues of Some Schrödinger Operators on Lattices

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ABSTRACT

The discrete Schrödinger operator $H_{\lambda\mu}$ on the subspace of even functions of the Hilbert space $\ell^2(\mathbb{Z}^n)$, with finite potential depending on $\lambda, \mu \in \mathbb{R}_{>0}$, is considered.

The dependence of the threshold resonance and eigenvalues on the parameters λ, μ and n are explicitly derived.

Keywords: Discrete Schrödinger operators, threshold resonance, eigenvalues, lattice

1. Introduction

In (Albeverio et al., 2006) an explicit example of a $-\Delta - V$ on the possesses both a threshold resonance and a threshold eigenvalue, where $-\Delta$ stands for the standard discrete Laplacian and V is a multiplication operator by the function $V(x) = \mu\delta_{x0} + \lambda\sum_{|s|=1}\delta_{xs}$, where $\lambda, \mu \in \mathbb{R}^2_{>0}$ and δ_{xs} is the Kroneker delta.

Beyond, the authors of (Lakaev and Bozorov, 2009) considered the restriction of this operator to the Hilbert space $\ell^2_e(\mathbb{Z}^3)$ of all even functions in $\ell^2_e(\mathbb{Z}^3)$. They investigated the dependence of the number of eigenvalues of $H_{\lambda\mu}$, on λ, μ ($\lambda > 0, \mu > 0$), and they showed that all eigenvalues arise either from a threshold resonance or from threshold eigenvalues under a variation of the interaction energy.

Moreover, they also proved that the first eigenvalue of the Hamiltonian H arises only from a threshold resonance under a variation of the interaction energy.

This result for the continuous two-particle Schrödinger operator was revealed by Newton (see p.1353 in (Newton, 1977)) and proved by Tamura (Tamura, 1993, Lemma 1.1) using a result by Simon (Simon, 1981).

In case $\lambda = 0$, Hiroshima et.al. (Hiroshima et al., 2012) showed that an embedded eigenvalue does appear for $n \geq 5$ but does not for $1 \leq n \leq 4$.

Our aim here is to investigate the spectrum of $H_{\lambda\mu}$, specifically, embedded eigenvalues and resonances at the edges of the continuous spectrum for any dimension $n \geq 1$.

2. The Dicrete Schrödinger Operator

2.1 The Discrete Laplacian

Let $\mathbb{Z}^n$ be the n–dimensional lattice, i.e. n–dimensional integer set. The Hilbert space of ℓ^2 sequences on $\mathbb{Z}^n$ is denoted by $\ell^2(\mathbb{Z}^n)$, and we use $\ell^2_e(\mathbb{Z}^n)$ to denote its subspace of all even functions.

On the Hilbert space $\ell^2_e(\mathbb{Z}^n)$, the discrete Laplacian Δ is usually associated with the following self-adjoint (bounded) multidimensional Toeplitz-type operator (see, e.g., (Mattis, 1986)):

$$\Delta = \frac{1}{2}\sum_{\substack{s\in\mathbb{Z}^n\\|s|=1}}(T(s) - T(0)),$$

where $T(y)$ is described as a sum of the two shift operators by y, and $-y$,

$y \in \mathbb{Z}^n$:

$$(T(y)f)(x) = \frac{1}{2}(f(x+y) + f(x-y)), \quad f \in \ell^2_e(\mathbb{Z}^n),\, x \in \mathbb{Z}^n.$$

Let a notation $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n = (-\pi, \pi]^n$ means the n-dimensional torus (the first Brillouin zone, i.e., the dual group of $\mathbb{Z}^n$) equipped its Haar measure, and let $L^2_e(\mathbb{T}^n)$ denote the subspace of all even functions of $L^2(\mathbb{T}^n)$-the Hilbert space of L^2 functions on $\mathbb{T}^n$.

The Laplacian Δ, in the momentum representation, i.e. in the Fourier representation, is introduced as

$$\widehat{\Delta} = \mathcal{F}^{-1}\Delta\mathcal{F},$$

where $\mathcal{F}$ stands for the standard Fourier transform $\mathcal{F} : L^2(\mathbb{T}^n) \longrightarrow \ell^2(\mathbb{Z}^n)$, and $\widehat{\Delta}$ acts as the multiplication operator

$$(\widehat{\Delta}f)(p) = -\mathfrak{e}(p)\hat{f}(p), \quad \hat{f} = \mathcal{F}f,\, p \in \mathbb{T}^n,$$

where

$$\mathfrak{e}(p) = \sum_{j=1}^{n}(1 - \cos p_j), \quad p \in \mathbb{T}^n.$$

In the physical literature, the function $\mathfrak{e}(\cdot)$ being a real valued-function on $\mathbb{T}^n$, is called the *dispersion relation* of the Laplace operator.

2.2 The Discrete Schrödinger Operator

The discrete Schrödinger operator in $\ell^2_e(\mathbb{Z}^n)$ is defined as

$$H_{\lambda\mu} = -\Delta - \widehat{V}(x),$$

where the potential $\widehat{V}(x)$ depends on two parameters $\lambda, \mu \in \mathbb{R}_{>0}$ and satisfies

$$\widehat{V}(x) = \begin{cases} \mu, & \text{if} \quad x = 0 \\ \lambda, & \text{if} \quad |x| = 1 \\ 0, & \text{if} \quad |x| > 0 \end{cases}, \quad x \in \mathbb{Z}^n,$$

which provides $H_{\lambda\mu}$ to be a bounded self-adjoint operator.

2.3 The Discrete Schrödinger Operator in Momentum Representation

The operator $H_{\lambda\mu}$ in the momentum representation acts in the Hilbert space $L_e^2(\mathbb{T}^n)$ as

$$H_{\lambda\mu} = H_0 - V,$$

where H_0 acts as the multiplication operator

$$(H_0 f)(p) = \mathfrak{e}(p) f(p), \quad f \in L_e^2(\mathbb{T}^n),\, p \in \mathbb{T}^n$$

and V is an integral operator convolution type

$$(Vf)(p) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{T}^n} v(p-s) f(s) ds, \quad f \in L_e^2(\mathbb{T}^n),\, p \in \mathbb{T}^n.$$

Here $v(\cdot)$ is the Fourier transform of $\widehat{V}(\cdot)$ computed as

$$v(p) = \frac{1}{(2\pi)^{\frac{n}{2}}} \left(\mu + \lambda \sum_{i=1}^{n} \cos p_i \right),$$

and it gives for the potential operator V the following representation

$$V = \mu \langle \cdot, \mathrm{c}_0 \rangle \mathrm{c}_0 + \frac{\lambda}{2} \sum_{j=1}^{n} \langle \cdot, \mathrm{c}_j \rangle \mathrm{c}_j,$$

where $\{\mathrm{c}_0, \mathrm{c}_j : j = 1, \ldots, n\}$ is the following orthonormal system in $L_e^2(\mathbb{T}^n)$

$$\mathrm{c}_0(p) = \frac{1}{(2\pi)^{\frac{n}{2}}} = \mathrm{const}, \quad \mathrm{c}_j(p) = \frac{\sqrt{2}}{(2\pi)^{\frac{n}{2}}} \cos p_j, \quad j = 1, \ldots, n,\, p \in \mathbb{T}^n,$$

and $\langle \cdot, \cdot \rangle$ means the inner product on $L_e^2(\mathbb{T}^n)$.

2.4 The Essential Spectrum

The perturbation V of the operator $H_{\lambda\mu}$ is a finite operator and, therefore, in accordance with the Weyl theorem on the stability of the essential spectrum the equality $\sigma_{ess}(H_{\lambda\mu}) = \sigma_{ess}(H_0)$ holds, and moreover $\sigma_{ess}(H_{\lambda\mu}) = \sigma(H_0)$, and hence the essential spectrum $\sigma_{ess}(H_{\lambda\mu})$ fills in the following interval on the real axis:

$$\sigma_{\mathrm{ess}}(H_{\lambda\mu}) = [\mathfrak{e}_m, \mathfrak{e}_M],$$

where

$$\mathfrak{e}_m = \min_{p \in \mathbb{T}^n} \mathfrak{e}(p) = 0, \quad \mathfrak{e}_M = \max_{p \in \mathbb{T}^n} \mathfrak{e}(p) = 2n.$$

Theorem 2.1. *The essential spectrum is a pure absolute continuous spectrum, i.e.* $\sigma_{ess}(H_{\lambda\mu}) = \sigma_{ac}(H_{\lambda\mu}) = [\mathfrak{e}_m, \mathfrak{e}_M]$.

Proof. For the proof see (Bellissard and Schulz-Baldes, 2012). □

3. The Birman-Schwinger Principle

The Birman-Schwinger principle allows us to reduce the problem to study of the compact (finite) operators.

Denote by $(H_0 - z)^{-1}$ the resolvent of H_0, where $z \in \mathbb{C} \setminus [\mathfrak{e}_m, \mathfrak{e}_M]$.

Let us write the following equality

$$(H_0 - z)^{-1} V_{\lambda\mu} = B_1 B_2, \tag{1}$$

where B_1, B_2 are vector valued operators defined by

$$B_1 = \big(\sqrt{\mu}(H_0 - z)^{-1/2}\mathrm{c}_0, \sqrt{\frac{\lambda}{2}}(H_0 - z)^{-1/2}\mathrm{c}_1, \ldots, \sqrt{\frac{\lambda}{2}}(H_0 - z)^{-1/2}\mathrm{c}_n\big) : \mathbb{C}^{n+1} \to L^2_e(\mathbb{T}^n),$$
$$(2)$$
$$B_2 = \big(\sqrt{\mu}\langle\cdot, (H_0 - z)^{-1/2}\mathrm{c}_0\rangle, \sqrt{\frac{\lambda}{2}}\langle\cdot, (H_0 - z)^{-1/2}\mathrm{c}_1\rangle, \ldots, \sqrt{\frac{\lambda}{2}}\langle\cdot, (H_0 - z)^{-1/2}\mathrm{c}_n\rangle\big)^T : L^2_e(\mathbb{T}^n) \to \mathbb{C}^{n+1}.$$

Note that $a_{ij}(z) := \langle (H_0 - z)^{-1}\mathrm{c}_j, \mathrm{c}_i\rangle$, $i, j = 0, 1, \ldots, n$, is a multiplication map on $\mathbb{C}$, and hence

$$G(z) = B_2 B_1 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$$

is described as an $(n+1) \times (n+1)$ matrix operator.

Lemma 3.1. *The number* $z \in \mathbb{C} \setminus [\mathfrak{e}_m, \mathfrak{e}_M]$ *is an eigenvalue of* $H_{\lambda\mu}$ *iff* $\nu = 1$ *is an eigenvalue of* $G(z)$.

Proof. The relation

$$Hf = zf \;\Leftrightarrow f = (H_0 - z)^{-1} V f \tag{3}$$

gives that the number $z \in \mathbb{C} \setminus [\mathfrak{e}_m, \mathfrak{e}_M]$ is an eigenvalue of H iff $\nu = 1$ is an eigenvalue of $(H_0 - z)^{-1}V$ in (1).

Due to spectrum of the product operators both operators $(H_0 - z)^{-1}V = B_1B_2$ and $G(z) = B_2B_1 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ have the same nonzero eigenvalues with the same multiplicities, a fact that completes the proof. □

Lemma 3.2. *Let $z \in \mathbb{C}\setminus[\mathfrak{e}_{\min}, \mathfrak{e}_{\max}]$. The vector $\vec{Z} = (w_0, w_1, \ldots, w_n) \in \mathbb{C}^{n+1}$, is an eigenvector of $G(z)$ associated to $\nu = 1$, iff $f = B_1\vec{Z}$, i.e.*

$$f(p) = \frac{(2\pi)^{-n}}{\mathfrak{e}(p) - z}\Big(\mu w_0 + \frac{\lambda}{\sqrt{2}}\sum_{j=1}^{n} w_j \cos p_j\Big) \tag{4}$$

is an eigenfunction of $H_{\lambda\mu}$ corresponding to z.

Proof. Due to spectrum of the product operators $G(z)\vec{Z} = \vec{Z}$, i.e. $B_2B_1\vec{Z} = \vec{Z}$ iff $f = (H_0 - z)^{-1}Vf = B_1B_2f$, where $f = B_1\vec{Z}$. Since (2), the function f coincides with (4). This fact together $f = (H_0 - z)^{-1}Vf$, i.e. $((H_0 - z) - V)f = 0$ ends the proof. □

Since $H_{\lambda\mu}$ is self-adjoint and V is positive, further it is enough to study the discrete spectrum $H_{\lambda\mu}$ in $(-\infty, \mathfrak{e}_m]$.

3.1 The Determinant of $G(z) - E_{n+1}$

Since the function $\mathfrak{e}(q) = \mathfrak{e}(q_1, \ldots, q_n)$ is invariant with respect to the permutations of its arguments $q_1, \ldots, q_n$, the integrals

$$a(z) := \langle \mathrm{c}_0, (H_0 - z)^{-1}\mathrm{c}_0\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{dq}{\mathfrak{e}(q) - z},$$

$$b(z) := \frac{1}{\sqrt{2}}\langle \mathrm{c}_0, (H_0 - z)^{-1}\mathrm{c}_j\rangle = \frac{1}{\sqrt{2}}\langle \mathrm{c}_j, (H_0 - z)^{-1}\mathrm{c}_0\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\cos q_j\, dq}{\mathfrak{e}(q) - z},$$
$$j = 1, \ldots, n$$

$$c(z) := \frac{1}{2}\langle \mathrm{c}_j, (H_0 - z)^{-1}\mathrm{c}_j\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\cos^2 q_j\, dq}{\mathfrak{e}(q) - z},$$
$$j = 1, \ldots, n,$$

$$d(z) := \frac{1}{2}\langle \mathrm{c}_i, (H_0 - z)^{-1}\mathrm{c}_j\rangle = \frac{1}{2}\langle \mathrm{c}_j, (H_0 - z)^{-1}\mathrm{c}_i\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\cos q_i \cos q_j\, dq}{\mathfrak{e}(q) - z},$$
$$i, j = 1 \ldots, n,\ i \neq j,$$

do not depend on the particular choice of the indices i, j.

From the definition of $G(z)$, it's coefficients g_{ij} are described as

$$g_{00}(z) = \mu a(z), \quad g_{0j}(z) = \frac{\lambda}{\sqrt{2}} b(z), \quad j = 1, \ldots, n,$$
$$g_{i0}(z) = \sqrt{2}\mu b(z), \quad g_{ii}(z) = \lambda c(z), \quad g_{ij}(z) = \lambda d(z), \quad j = 1, \ldots n,\ j \neq i,$$

Hence the matrix $G(z)$ has the form

$$G(z) = \begin{pmatrix} \mu a(z) & \frac{\lambda}{\sqrt{2}} b(z) & \ldots & \ldots & \frac{\lambda}{\sqrt{2}} b(z) \\ \sqrt{2}\mu b(z) & \lambda c(z) & \lambda d(z) & \ldots & \lambda d(z) \\ \vdots & \lambda d(z) & \ddots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \ddots & \lambda d(z) \\ \sqrt{2}\mu b(z) & \lambda d(z) & \ldots & \lambda d(z) & \lambda c(z) \end{pmatrix}. \tag{5}$$

Using the assertions on the calculation of determinants we take

$$\det(G(z) - E_{n+1}) = \delta_1(\lambda, \mu : z) \cdot \delta_0(\lambda : z),$$

where E_{n+1} is the identity $(n+1) \times (n+1)$ matrix and

$$\delta_1(\lambda, \mu : z) = (1 - \mu a(z))(1 - \lambda(c(z) + (n-1)d(z))) - n\mu\lambda b^2(z), \quad \delta_0(\lambda : z) = (\lambda(c(z) - d(z)) - 1)^{n-1}.$$

Lemma 3.3. *The number $z \in \mathbb{C} \setminus [\mathfrak{e}_m, \mathfrak{e}_M]$ is an eigenvalue of $H_{\lambda\mu}$ iff $\delta_1(\lambda, \mu : z) = 0$ or $\delta_0(\lambda : z) = 0$.*

Proof. This lemma is a corollary of Lemma 3.1. □

Let $N(z)$ be the number of eigenvalues of $H_{\lambda\mu}$ smaller than z, $z \leq \mathfrak{e}_{\mathrm{m}}$ counted with their multiplicities.

Now for self-adjoint upper bounded operator A in the abstract Hilbert space, we define $n(\nu, A)$ -the number of eigenvalues of A larger than ν (counted with their multiplicities), where $\nu > \sup \sigma_{ess}(A)$.

Lemma 3.4. *Let $z \leq \mathfrak{e}_{\mathrm{m}}$. Then*

$$N(z) = n(1, G(z)) \tag{6}$$

and

$$N(z) \leq n + 1. \tag{7}$$

Proof. The equality (6) follows using the variational principle.

The relation

$$\langle Hf, f\rangle < z\langle f, f\rangle \ \Leftrightarrow \langle g, g\rangle < \langle (H_0 - z)^{-1/2} V (H_0 - z)^{-1/2} g, g\rangle, \quad g = (H_0 - z)^{-1/2} f, \tag{8}$$

and $\mathrm{Ker}(H_0 - z) = \{0\}$ give that

$$N(z) = n(1, (H_0 - z)^{-1/2} V (H_0 - z)^{-1/2}).$$

Due to spectrum of the product operators both operators $(H_0 - z)^{-1}V = B_1 B_2$ and $G(z) = B_2 B_1 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ have the same nonzero eigenvalues with the same multiplicities, a fact that completes the proof of (6), where B_1, B_2 are vector valued operators defined by (2).

Since $G(z)$ has rank less than or equal $n + 1$ and (6), we get (7). □

4. Properties of $\det(G(z) - E_{n+1})$

Set

$$\alpha(z) := c(z) + (n - 1)d(z), \gamma(z) := a(z)(c(z) + (n - 1)d(z)) - nb^2(z) \tag{9}$$

Lemma 4.1. *For any $z < 0$ we have*

$$\begin{aligned} a(z) + b(z) &= \frac{1}{n} + \frac{z}{n} a(z), \\ \alpha(z) &= (n - z)b(z), \\ \gamma(z) &= b(z) \end{aligned}$$

Proof. See Appendix 1. □

4.1 Zeroes of $\delta_0(\lambda : z)$

Let us write $\delta_0(\lambda : z) = (\varrho_0(\lambda : z))^{n-1}$ where

$$\varrho_0(\lambda : z) = \lambda(c(z) - d(z)) - 1.$$

Set

$$\lambda_c = (c - d)^{-1},$$

where

$$c - d := \lim_{z \to 0-} c(z) - d(z).$$

Lemma 4.2. *(a) For any $\lambda \leq \lambda_c$ the function $\varrho_0(\lambda : \ldots)$ has no zero in $(-\infty, \mathfrak{e}_m)$.*

(a') If $\lambda = \lambda_c$ then $\varrho_0(\lambda : \mathfrak{e}_m) = 0$.

(b) For any $\lambda > \lambda_c$ the function $\varrho_0(\lambda : \ldots)$ has a unique zero in $(-\infty, \mathfrak{e}_m)$ with multiplicity one.

Proof. Since $\frac{\partial}{\partial z}\varrho_0(\lambda : z) > 0$, $z \in (-\infty, \mathfrak{e}_m)$, the function $\varrho_0(\lambda : \ldots)$ is strictly monotone increasing in $(-\infty, \mathfrak{e}_m)$.

Then $\varrho_0(\lambda : z) \leq \varrho_0(\lambda_c : z) < \varrho_0(\lambda_c : \mathfrak{e}_m) = 0$ proves (a) and (a').

b) Since $\varrho_0(\lambda : \mathfrak{e}_m) > \varrho_0(\lambda_c : \mathfrak{e}_m) = 0$ and $\lim_{z \to -\infty} \varrho_0(\lambda : z) = -1$ there exists zeros of $\varrho_0(\lambda : \cdot)$ in the interval $(-\infty, \mathfrak{e}_m)$

Due to monotonicity of $\varrho_0(\lambda : \cdot)$ this zero is a unique and has multiplicity one. □

Corollary 4.1. *(a) For any $\lambda \leq \lambda_c$ the function $\delta_0(\lambda : \ldots)$ has no zero in $(-\infty, \mathfrak{e}_m)$.*

(a') If $\lambda = \lambda_c$ then $\delta_0(\lambda : \mathfrak{e}_m) = 0$.

(b) For any $\lambda > \lambda_c$ the function $\delta_0(\lambda : \ldots)$ has a unique zero in $(-\infty, \mathfrak{e}_m)$ with multiplicity $n - 1$.

This corollary and (7) give

Corollary 4.2. *The function $\delta_1(\lambda, \mu : \cdot)$ may have at most two zeros.*

Proof. Since $\natural\{z \in (-\infty, \mathfrak{e}_m) : \delta_0(\lambda : z) = 0\} = 0$ or $\natural\{z \in (-\infty, \mathfrak{e}_m) : \delta_0(\lambda : z) = 0\} = n - 1$ and $N(z) <= n + 1$ we get the proof of the lemma. □

4.2 The zeros of $\delta_1(\lambda, \mu : z)$

4.2.1 Case $n \geq 3$

The $\delta_1(\lambda, \mu : z)$ is had the view

$$\delta_1(\lambda, \mu : z) = 1 - \mu a(z) - \lambda\alpha(z) + \lambda\mu\gamma(z)$$

Since $\mathfrak{e}(\cdot)$ has a unique non-degenerate minimum at the origin, in case $n \geq 3$, the integrals $a(z), b(z), c(z)$ and $d(z)$ have continuation at $z = \mathfrak{e}_m$, and we denote them a, b, c and d, respectively.

According to the two equalities Lemma 4.1 and (9) we have

$$\delta_1(\lambda, \mu : \mathfrak{e}_m) = 1 - \mu a - \lambda nb + \lambda\mu b = 0, \quad \delta_1(\lambda, \mu : \mathfrak{e}_m) = (1 - \mu a - \lambda\alpha + \lambda\mu\gamma)$$

which is hyperbola with asymptotic $\lambda = \frac{a}{b}$ and $\mu = n$ in the quarter $(\lambda, \mu) \in \mathbb{R}^2_{>0}$.

Then the brunches of this hyperbola

$$\partial G_0 = \{(\lambda, \mu) \in \mathbb{R}^2_{>0} : \delta_1(\lambda, \mu : \mathfrak{e}_m) = 0, \quad \lambda = \frac{a}{b}\},$$
$$\partial G_2 = \{(\lambda, \mu) \in \mathbb{R}^2_{>0} : \delta_1(\lambda, \mu : \mathfrak{e}_m) = 0, \quad \lambda = \frac{a}{b}\},$$

split $\mathbb{R}^2_{>0}$ into three areas

$$G_0 = \{(\lambda, \mu) \in \mathbb{R}^2_{>0} : \delta_1(\lambda, \mu : \mathfrak{e}_m) > 0, \quad \lambda < \frac{a}{b}\},$$
$$G_1 = \{(\lambda, \mu) \in \mathbb{R}^2_{>0} : \delta_1(\lambda, \mu : \mathfrak{e}_m) < 0\},$$
$$G_2 = \{(\lambda, \mu) \in \mathbb{R}^2_{>0} : \delta_1(\lambda, \mu : \mathfrak{e}_m) > 0, \quad \lambda > \frac{a}{b}\},$$

Let $1 - \mu a(\mathfrak{e}_m) < 0$ resp. $1 - \lambda\alpha(\mathfrak{e}_m) < 0$. Then as the proof of Lemma 4.2 we can show that there exist their unique zeroes in $(-\infty, \mathfrak{e}_m)$ of the functions $1 - \mu a(\cdot) < 0$ and $1 - \lambda\alpha(\cdot) < 0$, and we denote them as z_μ resp. z_λ.

Lemma 4.3. *(a) Let $(\lambda, \mu) \in G_0$. Then $\delta_1(\lambda, \mu : z)$ has no zero in $(-\infty, \mathfrak{e}_m)$.*
(b) Let $(\lambda, \mu) \in G_1$. Then $\delta_1(\lambda, \mu : z)$ has unique zero in $(-\infty, \mathfrak{e}_m)$.
(c) Let $(\lambda, \mu) \in G_2$. Then $\delta_1(\lambda, \mu : z)$ has two zeroes $z_1(\lambda, \mu)$ and $z_2(\lambda, \mu)$ in $(-\infty, \mathfrak{e}_m)$. Moreover $z_1(\lambda, \mu) < z_2(\lambda, \mu)$.

Proof. (a) Let $(\lambda, \mu) \in G_0$. Then according to the monotonicity of $a(z), \alpha(z), b(z)$ we get

$$1 - \mu a(z) > 1 - \mu a(\mathfrak{e}_m), \quad 1 - \lambda\alpha(z) > 1 - \lambda\alpha(\mathfrak{e}_m), \quad -\lambda\mu b^2(z) > -\lambda\mu b^2(\mathfrak{e}_m)$$

for any z in $(-\infty, \mathfrak{e}_m)$.

And hence

$$\delta_1(\lambda, \mu : z) = \big(1 - \mu a(z)\big)\big(1 - \lambda\alpha(z)\big) - \lambda\mu n b^2(z) >$$
$$\big(1 - \mu a(\mathfrak{e}_m)\big)\big(1 - \lambda\alpha(\mathfrak{e}_m)\big) - \lambda\mu b^2(\mathfrak{e}_m) = \delta_1(\lambda, \mu : \mathfrak{e}_m) = 0$$

Then according Lemma 3.3 the assertion a) is correct.

(b) Let $(\lambda, \mu) \in G_1$. Then $\delta_1(\lambda, \mu : \mathfrak{e}_m) < 0$ implies there exists z_0 in $(-\infty, \mathfrak{e}_m)$, such that $\delta_1(\lambda, \mu : z_0) = 0$.

In that case if z_0 is not unique then due to properties of analytic functions $\delta_1(\lambda, \mu : \cdot)$ has at lest three zeroes (with multiplicity). This fact is contradiction to Corollary 4.2, and hence z_0 is unique.

(c) Let $(\lambda, \mu) \in G_2$. Then $1 - \mu a(\mathfrak{e}_m) < 0$ and $1 - \lambda\alpha(\mathfrak{e}_m) < 0$.

Setting $\zeta_{\min} = \min\{z_\lambda, z_\mu\}$, $\zeta_{\max} = \max\{z_\lambda, z_\mu\}$ we see that $\delta_1(\lambda, \mu : \zeta_{\min}) = -\lambda\mu b^2(\zeta_{\min}) < 0$, $\delta_1(\lambda, \mu : \zeta_{\max}) = -\lambda\mu b^2(\zeta_{\max}) < 0$ which prove δ_1 has two zeros z_1 and z_2 satisfying

$$z_1 < \zeta_{\min} \le \zeta_{\max} < z_2 < \mathfrak{e}_m.$$

□

4.3 Case $n = 1, 2$

Using Lemma 4.1 we write

$$\delta_1(\lambda, \mu : z) = (-n\lambda - \mu + \lambda\mu)a(z) + 1 + \lambda - \frac{\lambda\mu}{n} + \big(\lambda(2z - \frac{z^2}{n}) - \frac{\lambda\mu}{n} z\big)a(z) - \frac{\lambda}{n} z. \tag{10}$$

In case $n = 1$. Elementary calculations give

$$a(z) = \frac{1}{\sqrt{-z}\sqrt{2 - z}}$$

and hence from 10 we get

Lemma 4.4. *For any $\mu, \lambda \geq 0$ the asymptotics*

$$\Delta_1(\mu, \lambda; z) = C_{-\frac{1}{2}}(\mu, \lambda)(-z)^{-\frac{1}{2}} + C_0(\mu, \lambda) + O((-z)^{\frac{1}{2}}), \quad z \to 0-, \qquad (11)$$

is valid, where

$$C_{-\frac{1}{2}}(\mu, \lambda) = \frac{\mu\lambda - (\mu + \lambda)}{\sqrt{2}}, \quad C_0(\mu, \lambda) = 1 - \lambda(\mu - 1).$$

This lemma helps to receive the following assertion

Proposition 4.1. *Let $\mu, \lambda > 0$. Further*

(a) *if $\mu\lambda < \mu + \lambda$, then $\lim_{z \to 0-} \Delta_1(\mu, \lambda; z) = -\infty$;*

(a′) *in case $\mu\lambda > \mu + \lambda$, we have $\lim_{z \to 0-} \Delta_1(\mu, \lambda; z) = +\infty$;*

(b) *when $\mu\lambda = \mu + \lambda$, the limit $\lim_{z \to 0-} \Delta_1(\mu, \lambda; z) = 1 - \mu < 0$ holds.*

In case $n = 2$**.** The asymptotics

$$a(z) = -\frac{\sqrt{2}}{2\pi} \ln(-z) + (\frac{1}{2} - \frac{\sqrt{2}}{\pi}) + O(-z),$$

can be found in (Lakaev and Tilovova, 1994), and since it's proof is long we refer to this paper for the proof.

The last asymptotics and (10) lead

Lemma 4.5. *Let $\lambda, \mu \geq 0$. Then*

$$\delta_1(\mu, \lambda; z) = C(\mu, \lambda) \ln(-z) + C_0(\mu, \lambda) + O(-z), \quad z \to 0-,$$

as $z \to 0-$, where

$$C(\mu, \lambda) = \frac{1}{\sqrt{2}\pi}\Big((\mu + 2\lambda) - \mu\lambda\Big), \quad C_0 = (\frac{1}{2} - \frac{\sqrt{2}}{\pi})\Big(-(\mu + 2\lambda) + \mu\lambda\Big) + 1 + \lambda + \frac{\lambda\mu}{2}$$

Hence we get

Proposition 4.2. *Let $\lambda, \mu \geq 0$. Then* *(a)* $\lim_{z\to 0-} \delta_1(\mu, \lambda; z) = -\infty$, *if* $\mu + 2\lambda - \mu\lambda < 0$

(b) $\lim_{z\to 0-} \delta_1(\mu, \lambda; z) = +\infty$, *if* $\mu + 2\lambda - \mu\lambda > 0$

(c) $\lim_{z\to 0-} \delta_1(\mu, \lambda; z) = 1 - 2\lambda < 0$, *if* $\mu + 2\lambda - \mu\lambda = 0$

We use the notation $P(\lambda, \mu)$ for hyperbolas $\mu + 2\lambda - \mu\lambda = 0$ when $n = 2$ and $\mu + \lambda - \mu\lambda = 0$ when $n = 1$.

Only one brunche of this hyperbola

$$\partial G_2 = \{(\lambda, \mu) \in \mathbb{R}^2_{>0} : P(\lambda, \mu) = 0\},$$

exists in $\mathbb{R}^2_{>0}$ and then we split $\mathbb{R}^2_{>0}$ into two areas

$$G_1 = \{(\lambda, \mu) \in \mathbb{R}^2_{>0} : P(\lambda, \mu) > 0\},$$
$$G_2 = \{(\lambda, \mu) \in \mathbb{R}^2_{>0} : P(\lambda, \mu) < 0\}.$$

We have the following lemma

Lemma 4.6. *Assume $n = 1, 2$. (a) Let $(\lambda, \mu) \in G_1 \cup \partial G_2$. Then $\delta_1(\lambda, \mu : z)$ has unique zero in $(-\infty, \mathfrak{e}_m)$.*

(b) Let $(\lambda, \mu) \in G_2$. Then $\delta_1(\lambda, \mu : z)$ has two zeroes $z_1(\lambda, \mu)$ and $z_2(\lambda, \mu)$ in $(-\infty, \mathfrak{e}_m)$. Moreover $z_1(\lambda, \mu) < z_2(\lambda, \mu)$.

Proof. The proof could be taken as Lemma 4.3. □

5. The View of Eigenfunctions

When $\delta_0(\lambda; z) = 0$ then the solutions of $G(z)u = u$, $u \in \mathbb{C}^{n+1}$, has form

$$u_1 = (0, 1, -1, 0, \ldots, 0), \quad u_2 = (0, 1, 0, -1, 0, \ldots, 0), \quad u_{n-1} = (0, 1, 0, \ldots, 0, 1)$$

and hence by Lemma 3.2 corresponding eigenfunctions of $H_{\lambda\mu}$ have the forms

$$g_j(p) = (H_0 - z)^{-1}(\cos p_1 - \cos p_j), \quad j = 2, \ldots, n.$$

Due to Lemma 4.3, the function $\delta_1(z)$ may have at most two zeroes in $(-\infty, \mathfrak{e}_m)$.

Without of loss generality, we assume z_1 and z_2 be zeroes of δ_1. Then the corresponding equation has form

$$u_i = \big(\frac{n\lambda}{\sqrt{2}} b(z_i)(1-\mu a(z_i))^{-1}, 1, \dots, 1\big), \quad i = 1, 2,$$

and by virtue of Lemma 3.2, corresponding eigenfunction of $H_{\lambda\mu}$ has the forms

$$g_i(p) = (H_0 - z)^{-1}\Big(\frac{n\lambda}{\sqrt{2}} b(z_i)(1-\mu a(z_i))^{-1} + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^{n} \cos p_j\Big), \quad i = 1, 2.$$

6. The Resonance and Embedded Eigenvalues

Definition 6.1. *If the solution of the equation $H_{\lambda\mu} f = \mathfrak{e}_m f$ belong to $\ell_e^2(\mathbb{Z}^n)$ (does not belong to $\ell_e^2(\mathbb{Z}^n)$) then we say that $H_{\lambda\mu}$ has threshold eigenvalues (threshold resonance).*

6.1 The Resonance and Embedded Eigenvalues Corresponding to δ_1

In case $n = 2$, the integrals $a(z), b(z), c(z)$ and $d(z)$ have no continuation at $z = \mathfrak{e}_m$, but we can define

$$c - d := \lim_{z \to \mathfrak{e}_m - 0} c(z) - d(z),$$

and then we get the continuation of δ_1 at $z = \mathfrak{e}_m$, when $n \geq 2$.

Using the similar procedure in Section 5 and Lemma 4.1 we get

Lemma 6.1. *Let $\lambda_c = (a - c)^{-1}$ then the threshold $\mathfrak{e}_m = 0$ is an eigenvalue of $H_{\lambda\mu}$ with eigenfunctions*

$$g(p) = \frac{\cos p_1 - \cos p_j}{\mathfrak{e}(p)}, \quad j = 2, \dots, n.$$

If $\lambda \neq \lambda_c$, the operator $H_{\lambda\mu}$ has no threshold resonance and embedded eigenvalue.

6.2 The Resonance and Embedded Eigenvalues corresponding to δ_1

Since Lemmas 4.1, 4.2 the function δ_1 has no continuation at $z = \mathfrak{e}_m$, when $n = 1, 2$.

Lemma 6.2. *Let $\delta_1(\lambda,\mu;0)=0$. Then $H_{\lambda\mu}$ has a threshold resonance (embedded eigenvalue with multiplicity one) if $n=3,4$ ($n\geq 5$) with eigenvector*

$$g_{\lambda,\mu}(p)=\frac{1}{\mathfrak{e}(p)}\Phi_i(p),\quad \Phi(p)=\frac{n\lambda}{\sqrt{2}}b(1-\mu a)^{-1}+\frac{\lambda}{\sqrt{2}}\sum_{j=1}^{n}\cos p_j. \tag{12}$$

Proof. Let

$$\delta_1(\lambda,\mu;0)=0.$$

Using the similar procedure in Section 5 and Lemma 4.3 we get $H_{\lambda\mu}f=0$ has a solution having view (12).

Since $\mu=n$ is asymptotics of the hyperbola $\delta_1(\lambda,\mu:\mathfrak{e}_m)=0$ we have

$$\Phi(0)=\frac{n\lambda}{\sqrt{2}3}b(1-\mu a)^{-1}(n-\mu)\neq 0.$$

Then due to $\int_{\mathbb{T}^n}\frac{dp}{\mathfrak{e}^2(p)}=\infty$ as $n=3,4$ and $\int_{\mathbb{T}^n}\frac{dp}{\mathfrak{e}^2(p)}<\infty$ as $n\geq 5$ the eigenfunction $g_{\lambda,\mu}(p)$ does not belong to $L_e^2(\mathbb{T}^n)$, but does to $L_e^1(\mathbb{T}^n)$, as $n=3,4$, while it belongs to $L_e^2(\mathbb{T}^n)$ as $n\geq 5$. □

7. Main Theorem

Note that all the theorems in this section are derived from Corollary 4.1 and Lemmas 4.3, 4.6, 6.1 and 12.

Introduce half planes and their boundary

$$G_c^l=\{(\lambda,\mu)\in\mathbb{R}^2_{>0}:\lambda<\lambda_c\},\quad G_c^r=\{(\lambda,\mu)\in\mathbb{R}^2_{>0}:\lambda>\lambda_c\},$$
$$\partial G_c=\{(\lambda,\mu)\in\mathbb{R}^2_{>0}:\lambda=\lambda_c\},$$

and set

$$D_0=G_0,\quad D_1=G_1\cap G_c^l,\quad D_2=G_2\cap G_c^l,$$
$$D_{n+1}=G_1\cap G_c^r,\quad D_{n+2}=G_2\cap G_c^r.$$

The sets G_0,G_2, ∂G_0,∂G_2 G_c^l,G_c^r create non intersecting five lines such that

$$B_0=\partial G_0,\quad B_1=\partial G_2\cap G_c^l,\quad B_n=\partial G_2\cap G_c^r,$$
$$E_1=\partial G_c\cap G_1,\quad E_2=\partial G_c\cap G_2,$$

and one point set

$$E = \partial G_2 \cap \partial G_c.$$

And the union of these sets are equal to $\partial G_2 \cup \partial G_c$.

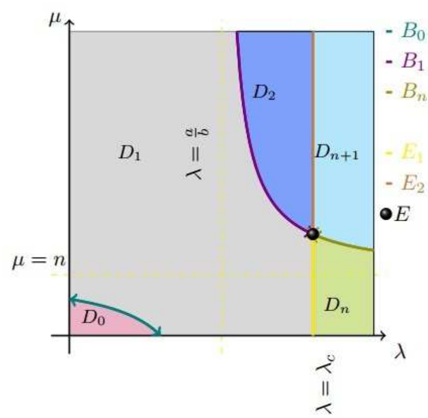

Figure 1: Case $n \geq 3$

Theorem 7.1. *Let $n \geq 3$. a) Assume $(\lambda, \mu) \in D_k$, $k \in \{0, 1, 2, n, n+1\}$, then $H_{\lambda\mu}$ has k eigenvalues below the essential spectrum.*

b) Assume $(\lambda, \mu) \in B_k$, $k \in \{0, 1, n\}$ and $n = 3, 4$ $(n \geq 5)$. Then $\mathfrak{e}_m$ is a threshold resonance (embedded eigenvalue with multiplicity one) and $H_{\lambda\mu}$ has k eigenvalues below the essential spectrum.

c) Assume $(\lambda, \mu) \in E_k$, $k \in \{1, 2\}$ and $n \geq 3$. Then $\mathfrak{e}_m$ is a embedded eigenvalue with multiplicity $n-1$ and $H_{\lambda\mu}$ has no threshold resonance and has k eigenvalues below the essential spectrum.

d) Assume $(\lambda, \mu) \in E$ and $n = 3, 4$ $(n \geq 5)$. Then $\mathfrak{e}_m$ is a threshold resonance and embedded eigenvalue with multiplicity $n-1$ (embedded eigenvalue with multiplicity n) and $H_{\lambda\mu}$ has one eigenvalues below the essential spectrum.

7.1 Case $n = 2$

We know in case $n = 2$ the sets G_c^l, G_c^r exists while this type sets do not in case $n = 1$, and set

$$D_1 = G_1 \cup \partial G_2 \cap G_c^l, \quad D_2 = G_2 \cap G_c^l, \quad D_2 = G_1 \cap G_c^r, \quad D_3 = G_2 \cap G_c^r,$$

and non intersecting two lines

$$E_1 = \partial G_c \cap (G_1 \cup \partial G_2), \quad E_2 = \partial G_c \cap G_2,$$

and one point set

$$E = \partial G_2 \cap \partial G_c.$$

And the union of the last sets are equal to $\partial G_2 \cup \partial G_c$.

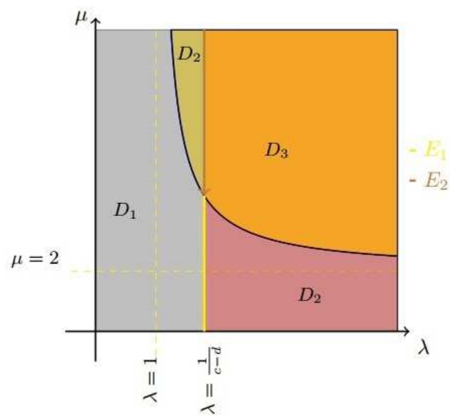

Figure 2: Case $n = 2$

Theorem 7.2. *(a) Assume $(\lambda, \mu) \in D_k$, $k \in \{1, 2, 3\}$, then $H_{\lambda\mu}$ has k eigenvalues below the essential spectrum.*
(b) Assume $(\lambda, \mu) \in E_k$, $k \in \{1, 2\}$. Then $\mathfrak{e}_m$ is a embedded eigenvalue with multiplicity 1 and $H_{\lambda\mu}$ has no threshold eigenvalue and has k eigenvalues below the essential spectrum.

7.2 Case $n = 1$

Set

$$D_1 = G_1 \cup \partial G_2, \quad D_2 = G_2.$$

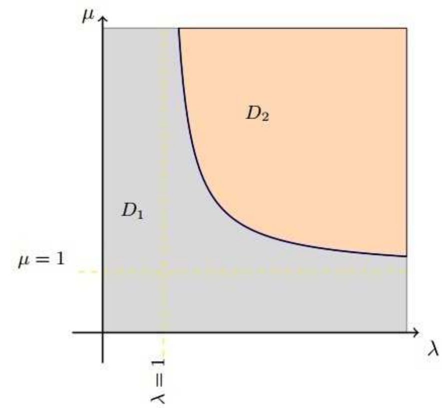

Figure 3: Case $n = 1$

Theorem 7.3. *Assume $(\lambda, \mu) \in D_k$, $k \in \{1, 2\}$, then $H_{\lambda\mu}$ has k eigenvalues below the essential spectrum, and moreover $H_{\lambda\mu}$ has no threshold resonance and embedded eigenvalue.*

7.3 The proof of Lemma 4.1

$$a(z) - b(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{(1 - \cos q_1)dq}{\mathfrak{e}(q) - z} = \frac{1}{n} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\sum_{j=1}^n (1 - \cos q_j)dq}{\mathfrak{e}(q) - z} =$$
$$\frac{1}{n} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{(\mathfrak{e}(z) - z + z)dq}{\mathfrak{e}(q) - z} = \frac{1}{n} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} dq + \frac{z}{n} \int_{\mathbb{T}^n} \frac{dq}{\mathfrak{e}(q) - z} = \frac{1}{n} + \frac{z}{n} a(z);$$

$$c(z) + (n-1)(z)d = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\cos q_1 \sum_{j=1}^n \cos q_j dq}{\mathfrak{e}(q) - \mathfrak{e}} =$$
$$\frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\cos q_1 (z - \mathfrak{e}(z))dq}{\mathfrak{e}(q) - \mathfrak{e}} + \frac{n - z}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\cos q_1 \sum_{j=1}^n 1 dq}{\mathfrak{e}(q) - \mathfrak{e}} = (n - z)b(z)$$

From the last equalities we get the proof of third equality of the lemma.

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